

Borel Reductions and Cub Games

Vadim Kulikov
Department of Mathematics and Statistics
University of Helsinki

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Abstract

It is shown that the power set of κ ordered by the subset relation modulo various versions of the non-stationary ideal can be embedded into the partial order of Borel equivalence relations on 2^κ under Borel reducibility. Here κ is uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$.

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1 Introduction

It is shown that the partial order of Borel equivalence relations on the generalized Baire spaces (2^κ for $\kappa > \omega$) under Borel reducibility has high complexity already at low levels (below E_0).

This extends an answer stated in [4] to an open problem stated in [5] and in particular solves open problems 7 and 9 from [4].

The development of the theory of the generalized Baire and Cantor spaces dates back to 1990's when it A. Mekler and J. Väänänen published the paper *Trees and Π_1^1 -Subsets of ${}^{\omega_1}\omega_1$* [13] and A. Halko published *Negligible subsets of the generalized Baire space $\omega_1^{\omega_1}$* . More recently equivalence relations and Borel reducibility on these spaces and their applications to model theory have been under focus, see my latest joint work with S. Friedman and T. Hyttinen [?].

Suppose κ is an infinite cardinal and let \mathcal{E}_κ^B be the collection of all Borel equivalence relations on 2^κ . (For definitions in the case $\kappa > \omega$ see next section.) For equivalence relations E_0 and E_1 let us denote $E_0 \leq_B E_1$ if there exists a Borel function $f: 2^\kappa \rightarrow 2^\kappa$ such that $(\eta, \xi) \in E_0 \iff (f(\eta), f(\xi)) \in E_1$. The relation \leq_B defines a quasiorder on \mathcal{E}_κ^B , i.e. it induces a partial order on $\mathcal{E}_\kappa^B / \sim_B$ where \sim_B is the equivalence relation of bireducibility: $E_0 \sim_B E_1 \iff (E_0 \leq_B E_1) \wedge (E_1 \leq_B E_0)$.

In the case $\kappa = \omega$ there are many known results that describe the order $\langle \mathcal{E}_\omega^B, \leq_B \rangle$. Two of them are:

Theorem (Louveau-Velickovic [12]). *The partial order $\langle \mathcal{P}(\omega), \subset_* \rangle$ can be embedded into the partial order $\langle \mathcal{E}_\omega^B, \leq_B \rangle$, where $A \subset_* B$ if $A \setminus B$ is finite.*

Theorem (Adams-Kechris [1]). *The partial order $\langle \mathcal{B}, \subset \rangle$ can be embedded into the partial order $\langle \mathcal{E}_\omega^B, \leq_B \rangle$, where \mathcal{B} is the collection of all Borel subsets of the real line \mathbb{R} . In fact, the embedding is into the suborder of $\langle \mathcal{E}_\omega^B, \leq_B \rangle$ consisting of the countable Borel equivalence relations, i.e., those Borel equivalence relations each of whose equivalence classes is countable.*

Our aim is to generalize these results to uncountable κ with $\kappa^{<\kappa} = \kappa$ and it is proved that $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\omega)} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$, where $A \subset_{\text{NS}(\omega)} B$ means that $A \setminus B$ is not ω -stationary. This is proved in ZFC. However under mild additional assumptions on κ or on the underlying set theory, it is shown that $\langle \mathcal{P}(\kappa), \subset_{\text{NS}} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$, where $A \subset_{\text{NS}} B$ means that $A \setminus B$ is non-stationary and that $\langle \mathcal{P}(\kappa), \subset_* \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$, where $A \subset_* B$ means that $A \setminus B$ is bounded.

Assumption. Everywhere in this article it is assumed that κ is a cardinal which satisfies $|\kappa^\alpha| = \kappa$ for all $\alpha < \kappa$. This requirement is briefly denoted by $\kappa^{<\kappa} = \kappa$.

2 Background in Generalized Descriptive Set Theory

1 Definition. Consider the function space 2^κ (all functions from κ to $\{0, 1\}$) equipped with the topology generated by the sets

$$N_p = \{\eta \in 2^\kappa \mid \eta \upharpoonright \alpha = p\}$$

for $\alpha < \kappa$ and $p \in 2^\alpha$. Borel sets on this space are obtained by closing the topology under unions and intersections of length $\leq \kappa$, and complements.

An equivalence relation E on 2^κ is *Borel reducible* to an equivalence relation E' on 2^κ if there exists a Borel function $f: 2^\kappa \rightarrow 2^\kappa$ (inverse images of open sets are Borel) such that $\eta E \xi \iff f(\eta) E' f(\xi)$. This is denoted by $E \leq_B E'$.

The descriptive set theory of these spaces, of equivalence relations on them and of their reducibility properties for $\kappa > \omega$, has been developed at least in [5, 7, 13]. For $\kappa = \omega$ this is the field of standard descriptive set theory.

By id_X we denote the identity relation on X : $(\eta, \xi) \in \text{id}_X \iff (\eta, \xi) \in X^2 \wedge \eta = \xi$ and by E_0 the equivalence relation on 2^κ (or on κ^κ as in the proof of Theorem 30) such that $(\eta, \xi) \in E_0 \iff \{\alpha \mid \eta(\alpha) \neq \xi(\alpha)\}$ is bounded.

Notation. Let \mathcal{E}_κ^B denote the set of all Borel equivalence relations on 2^κ (i.e. equivalence relations $E \subset (2^\kappa)^2$ such that E is a Borel set). If $X, Y \subset \kappa$ and $X \setminus Y$ is non-stationary, let us denote it by $X \subset_{\text{NS}} Y$. If $X \setminus Y$ is not λ -stationary for some regular $\lambda < \kappa$, it is denoted by $X \subset_{\text{NS}(\lambda)} Y$.

The set of all ordinals below κ which have cofinality λ is denoted by S_λ^κ , and $\text{lim}(\kappa)$ denotes the set of all limit ordinals below κ . Also $\text{reg } \kappa$ denotes the set of regular cardinals below κ and

$$S_{\geq \lambda}^\kappa = \bigcup_{\substack{\mu \geq \lambda \\ \mu \in \text{reg } \kappa}} S_\mu^\kappa,$$

$$S_{\leq \lambda}^\kappa = \bigcup_{\substack{\mu \leq \lambda \\ \mu \in \text{reg } \kappa}} S_\mu^\kappa.$$

If $A \subset \alpha$ and α is an ordinal, then $\text{OTP}(A)$ is the order type of A in the ordering induced on it by α .

For ordinals $\alpha < \beta$ let us adopt the following abbreviations:

- $(\alpha, \beta) = \{\gamma \mid \alpha < \gamma < \beta\},$
- $[\alpha, \beta] = \{\gamma \mid \alpha \leq \gamma \leq \beta\},$
- $(\alpha, \beta] = \{\gamma \mid \alpha < \gamma \leq \beta\},$
- $[\alpha, \beta) = \{\gamma \mid \alpha \leq \gamma < \beta\}.$

If η and ξ are functions in 2^κ , then $\eta \triangle \xi$ is the function $\zeta \in 2^\kappa$ such that $\zeta(\alpha) = 1 \iff \eta(\alpha) \neq \xi(\alpha)$ for all $\alpha < \kappa$, and $\bar{\eta} = 1 - \eta$ is the function $\zeta \in 2^\kappa$ such that $\zeta(\alpha) = 1 - \eta(\alpha)$ for all $\alpha < \kappa$. If A and B are sets, then $A \triangle B$ is just the symmetric difference.

For any set X , 2^X denotes the set of all functions from X to $2 = \{0, 1\}$. If $p \in 2^{[0, \alpha)}$ and $\eta \in 2^{[\alpha, \kappa)}$, then $p \frown \eta \in 2^\kappa$ is the catenation: $(p \frown \eta)(\beta) = p(\beta)$ for $\beta < \alpha$ and $(p \frown \eta)(\beta) = \eta(\beta)$ for $\beta \geq \alpha$.

2 Definition. A *co-meager* subset of X is a set which contains an intersection of length $\leq \kappa$ of dense open subsets of X . Co-meager sets are always non-empty and form a filter on 2^κ , [13]. A set X has the *Property of Baire* if there exists an open set A such that $X \triangle A$ is meager, i.e. a complement of a co-meager set. As in standard descriptive set theory, Borel sets have the Property of Baire (proved in [7]). For a Borel function $f: 2^\kappa \rightarrow 2^\kappa$ denote by $C(f)$ one of the co-meager sets restricted to which f is continuous (such set is not unique, but we can always pick one using the Property of Baire of Borel sets, see [5]).

3 Lemma. Let D be a co-meager set in 2^κ and let $p, q \in 2^\alpha$ for some $\alpha < \kappa$. Then there exists $\eta \in 2^{[\alpha, \kappa)}$ such that $p \frown \eta \in D$ and $q \frown \eta \in D$. Also there exists $\bar{\eta} \in 2^{[\alpha, \kappa)}$ such that $p \frown \bar{\eta} \in D$ and $q \frown \bar{\eta} \in D$ where $\bar{\eta} = 1 - \eta$.

Proof. Let h be the homeomorphism $N_p \rightarrow N_q$ defined by $p \frown \eta \mapsto q \frown \eta$. Then $h[N_p \cap D]$ is co-meager in N_q , so $N_q \cap D \cap h[N_p \cap D]$ is non-empty. Pick η' from that intersection and let $\eta = \eta' \upharpoonright [\alpha, \kappa)$. This will do. For the second part take for h the homeomorphism defined by $p \frown \eta \mapsto q \frown \bar{\eta}$. \square

3 On Cub-games and GC_λ -characterization

The notion of cub-games is a useful way to treat certain properties of subsets of cardinals. They generalize closed unbounded sets and are related to combinatorial principles such as \square_κ . Under mild set theoretic assumptions, they give characterizations of CUB-filters in different cofinalities. Treatments of this subject can be found for example in [8, 9, 10].

4 Definition. Let $A \subset \kappa$. The game $\text{GC}_\lambda(A)$ is played between players **I** and **II** as follows. There are λ moves and at the i :th move player **I** picks an ordinal α_i which is greater than any ordinal picked earlier in the game and then **II** picks an ordinal $\beta_i > \alpha_i$. Player **II** wins if $\sup_{i < \lambda} \alpha_i \in A$. Otherwise player **I** wins.

5 Definition. A set $C \subset \kappa$ is λ -closed for a regular cardinal $\lambda < \kappa$, if for all increasing sequences $\langle \alpha_i \in C \mid i < \lambda \rangle$, the limit $\sup_{i < \lambda} \alpha_i$ is in C . A set $C \subset \kappa$ is *closed* if it is λ -closed for all regular $\lambda < \kappa$. A set is λ -cub if it is λ -closed and unbounded and *cub*, if it is closed and unbounded. A set is λ -stationary, if it intersects all λ -cub sets and *stationary* if it intersects all cub sets.

6 Definition. We say that GC_λ -characterization holds for κ , if

$$\{A \subset \kappa \mid \mathbf{II} \text{ has a winning strategy in } GC_\lambda(A)\} = \{A \subset \kappa \mid A \text{ contains a } \lambda\text{-cub set}\}$$

and we say that GC -characterization holds for κ if GC_λ -characterization holds for κ for all regular $\lambda < \kappa$.

7 Definition. Assume $\kappa = \lambda^+$ and $\mu \leq \lambda$ a regular uncountable cardinal. The *square principle on κ for μ* , denoted \square_μ^κ , defined by Jensen in case $\lambda = \mu$, is the statement that there exists a sequence $\langle C_\alpha \mid \alpha \in S_{\leq \mu}^\kappa \rangle$ with the following properties:

1. $C_\alpha \subset \alpha$ is closed and unbounded in α ,
2. if $\beta \in \lim C_\alpha$, then $C_\beta = \beta \cap C_\alpha$,
3. if $\text{cf}(\alpha) < \mu$, then $|C_\alpha| < \mu$.

8 Remark. For $\omega < \mu < \lambda$ in the definition above, it was proved by Shelah in [14] that \square_μ^κ holds (can be proved in ZFC, for a proof see also [2, Lemma 7.7]). If $\mu = \lambda$, then $\square_\mu^\kappa = \square_\mu^{\mu^+}$ is denoted by \square_μ and can be easily forced or, on the other hand, it holds, if $V = L$. The failure of \square_μ implies that μ^+ is Mahlo in L , as pointed out by Jensen, see [11].

9 Definition. For $\kappa > \omega$, the set $I[\kappa]$ consists of those $S \subset \kappa$ that have the following property: there exists a cub set C and a sequence $\langle \mathcal{D}_\alpha \mid \alpha < \kappa \rangle$ such that

1. $\mathcal{D}_\alpha \subset \mathcal{P}(\alpha)$, $|\mathcal{D}_\alpha| < \kappa$,
2. $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$ for all $\alpha < \beta$,
3. for all $\alpha \in C \cap S$ there exists $E \subset \alpha$ unbounded in α and of order type $\text{cf}(\alpha)$ such that for all $\beta < \alpha$, $E \cap \beta \in \mathcal{D}_\gamma$ for some $\gamma < \alpha$.

10 Remark. The following is known.

1. $I[\kappa]$ is a normal ideal and contains the non-stationary sets.
2. If $\lambda < \kappa$ is regular and $S_\lambda^\kappa \in I[\kappa]$, then GC_λ -characterization holds for κ .
3. If μ is regular and $\kappa = \mu^+$, then $S_{< \mu}^\kappa \in I[\kappa]$, [14]. This follows also from 4. and Remark 8
4. When $\lambda > \omega$, then \square_λ^κ implies that $S_\lambda^\kappa \in I[\kappa]$ (take $\mathcal{D}_\alpha = \{C_\alpha \cap \beta \mid \beta < \alpha\}$).
5. $S_\omega^\kappa \in I[\kappa]$.
6. If $\kappa^{< \lambda} = \kappa = \lambda^+$, then GC_λ -characterization holds for κ if and only if $\kappa \in I[\kappa]$ if and only if $S_\lambda^\kappa \in I[\kappa]$, see [8, Corollary 2.4] and [14].

7. The existence of $\lambda < \kappa$ such that GC_λ -characterization does not hold for κ is equiconsistent with the existence of a Mahlo cardinal.¹ Briefly this is because the failure of the characterization implies the failure of \square_λ which implies that λ^+ is Mahlo in L as discussed above. On the other hand, in the Mitchell model, obtained from $S_{\text{in}} = \{\delta < \lambda \mid \delta \text{ is inaccessible}\}$ where $\lambda > \kappa$ is Mahlo, it holds that $S_{\text{in}} \notin I[\kappa^+]$, [8, Lemma 2.6].
8. If κ is regular and for all regular $\mu < \kappa$ we have $\mu^{<\lambda} < \kappa$, then $\kappa \in I[\kappa]$.

Remark. As Remark 10 shows, the assumption that GC_λ -characterization holds for κ is quite weak. For instance GC_ω -characterization holds for all regular $\kappa > \omega$ and GCH implies that GC_λ -characterization holds for κ for all regular $\lambda < \kappa$.

4 Main Results

Theorems 11 and 12 constitute the goal of this work. They are stated below but proved in the end of this section, starting at pages 13 and 15 respectively.

11 Theorem. *Assume that $\lambda < \kappa$ are regular and GC_λ -characterization holds for κ . Then the order $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\lambda)} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$ strictly between id_{2^κ} and E_0 . More precisely there exists a one-to-one map $F: \mathcal{P}(\kappa) \rightarrow \mathcal{E}_\kappa^B$ such that for all $X, Y \in \mathcal{P}(\kappa)$ we have $\text{id}_{2^\kappa} \prec_B F(X) \prec_B E_0$ and*

$$X \subset_{\text{NS}(\lambda)} Y \iff F(X) \leq_B F(Y).$$

12 Theorem. *Assume either $\kappa = \omega_1$ or $\kappa = \lambda^+ > \omega_1$ and \square_λ . Then the partial order $\langle \mathcal{P}(\kappa), \subset_{\text{NS}} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$.*

4.1 Corollaries

13 Corollary. *Assume that $\lambda < \kappa$ is regular. Additionally assume one of the following:*

1. $\kappa = \mu^+$, μ is regular and $\lambda < \mu$,
2. $\kappa = \lambda^+$ and \square_λ holds,
3. for all regular $\mu < \kappa$, $\mu^{<\lambda} < \kappa$ (e.g. κ is ω_1 or inaccessible).

Then the partial order $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\lambda)} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$.

Proof. Any of the assumptions 1 – 4 is sufficient to obtain GC_λ -characterization for κ by Remarks 10 and 8, so the result follows from Theorem 11. \square

14 Corollary. *The partial order $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\omega)} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$. In particular $\langle \mathcal{P}(\omega_1), \subset_{\text{NS}} \rangle$ can be embedded into $\langle \mathcal{E}_{\omega_1}^B, \leq_B \rangle$ assuming CH.*

Proof. By Remark 10 GC_ω -characterization holds for κ for any regular $\kappa > \omega$, so the result follows from Theorem 11. \square

¹A good exposition of this result can be found in Lauri Tuomi's Master's thesis (University of Helsinki, 2009).

15 Definition. Let $S \subset \kappa$. Then the combinatorial principle $\diamond_\kappa(S)$ states that there exists a sequence $\langle D_\alpha \mid \alpha \in S \rangle$ such that for every $A \subset \kappa$ the set $\{\alpha \mid A \cap \alpha = D_\alpha\}$ is stationary.

16 Theorem (Shelah [15]). *If $\kappa = \lambda^+ = 2^\lambda$ and $S \subset \kappa \setminus S_{\text{cf}(\lambda)}^\kappa$ is stationary, then $\diamond_\kappa(S)$ holds.* \square

17 Corollary. 1. *The ordering $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$.*

2. *Assume that $\kappa = \omega_1$ and \diamond_{ω_1} holds or that κ is not a successor of an ω -cofinal cardinal. Then also the ordering $\langle \mathcal{P}(\kappa), \subset_* \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$, where \subset_* is inclusion modulo bounded sets.*

Proof. For the first part it is sufficient to show that the partial order $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\omega)} \rangle$. Let $G(A) = \bigcup_{i \in A} S_i$ where $\{S_i \subset S_\omega^\kappa \mid i < \kappa\}$ is a collection of disjoint stationary sets. Then $A \subset B \iff G(A) \subset_{\text{NS}} G(B)$, so this proves the first part.

For the second part, let us show that if $\diamond_\kappa(S_\lambda^\kappa)$ holds, then $\langle \mathcal{P}(\kappa), \subset_* \rangle$ can be embedded into $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\lambda)} \rangle$. Then the result follows. If $\kappa = \omega_1$ and \diamond_{ω_1} holds, then it follows by Corollary 14. On the other hand, if κ is not a successor of an ω -cofinal cardinal, then from Theorem 16 it follows that $\diamond_\kappa(S_\omega^\kappa)$ holds and the result follows again from Corollary 14.

Suppose that $\langle D_\alpha \mid \alpha \in S_\lambda^\kappa \rangle$ is a $\diamond_\kappa(S_\lambda^\kappa)$ -sequence. If $X, Y \subset \alpha$ for $\alpha \leq \kappa$, let $X \subset_* Y$ denote that there is $\beta < \alpha$ such that $X \setminus \beta \subset Y \setminus \beta$, i.e. X is a subset of Y on a final segment of α . Note that this coincides with the earlier defined \subset_* when $\alpha = \kappa$. For $A \subset \kappa$ let

$$H(A) = \{\alpha < \kappa \mid D_\alpha \subset_* A \cap \alpha\}.$$

If $A \subset_* B$ then there is $\gamma < \kappa$ such that $A \setminus \gamma \subset B \setminus \gamma$ and if $\beta > \gamma$ is in $H(A)$, then $D_\beta \subset_* A \cap \beta$ and since $A \cap \beta \subset_* B \cap \beta$, we have $D_\beta \subset_* B \cap \beta$, so $H(A) \subset_* H(B)$ which finally implies $H(A) \subset_{\text{NS}(\omega)} H(B)$.

Assume now that $A \not\subset_* B$ and let $C = A \setminus B$. Let S' be the stationary set such that for all $\alpha \in S'$, $C \cap \alpha = D_\alpha$. Let S be the λ -stationary set $S' \cap \{\alpha \mid C \text{ is unbounded below } \alpha\}$. S is stationary, because it is the intersection of S' and a cub set. Now for all $\alpha \in S$ we have $D_\alpha = C \cap \alpha \subset A \cap \alpha$, so $S \subset H(A)$. On the other hand if $\alpha \in S$, then

$$D_\alpha \setminus (B \cap \alpha) = (C \cap \alpha) \setminus (B \cap \alpha) = ((A \setminus B) \cap \alpha) \setminus (B \cap \alpha) = C \cap \alpha$$

is unbounded in α , so $D_\alpha \not\subset_* B \cap \alpha$ and so $S \subset H(A) \setminus H(B)$, whence $H(A) \not\subset_{\text{NS}(\lambda)} H(B)$. \square

18 Corollary. *There are 2^κ equivalence relations between id and E_0 that form a linear order with respect to \leq_B .*

Proof. Let $K = \{\eta \in 2^\kappa \mid (\exists \beta)(\forall \gamma > \beta)(\eta(\gamma) = 0)\}$, let $f: K \rightarrow \kappa$ be a bijection and for $\eta, \xi \in 2^\kappa$ define $\eta < \xi$ if and only if

$$\eta(\min\{\alpha \mid \eta(\alpha) \neq \xi(\alpha)\}) < \xi(\min\{\alpha \mid \eta(\alpha) \neq \xi(\alpha)\}).$$

For $\eta \in 2^\kappa$ let $A_\eta = \{f(\xi) \mid \xi < \eta \wedge \xi \in K\}$. Clearly $A_\eta \subsetneq A_\xi$ if and only if $\eta < \xi$ and the latter is a linear order. The statement now follows from Corollary 17. \square

4.2 Preparing for the Proofs

19 Definition. For each $S \subset \lim \kappa$ let us define equivalence relations E_S^* , E_S and $E_S^*(\alpha)$, $\alpha \leq \kappa$, on the space 2^κ as follows. Suppose $\eta, \xi \in 2^\delta$ for some $\delta \leq \kappa$ and let $\zeta = \eta \triangle \xi$. Let us define η and ξ to be $E_S^*(\delta)$ -equivalent if and only if for all ordinals $\alpha \in S \cap \delta$ there exists $\beta < \alpha$ such that $\zeta(\gamma)$ has the same value for all $\gamma \in (\beta, \alpha)$. Let $E_S^* = E_S^*(\kappa)$ and $E_S = E_S^* \cap E_0$, where E_0 is the equivalence modulo bounded sets.

Remark. If $S = \emptyset$, then $E_S = E_\emptyset = E_0$. If $S = \lim \kappa$ or equivalently if $S = \lim_\omega \kappa = S_\omega^\kappa$ (ω -cofinal limit ordinals), then $E_S = E_0'$, where E_0' is defined in [4].

20 Theorem. For any $S \subset \lim \kappa$ the equivalence relations E_S and E_S^* are Borel.

Proof. This is obvious by writing out the definitions:

$$\begin{aligned} E_S^* &= \bigcap_{\alpha \in S} \bigcup_{\beta < \alpha} \left(\bigcap_{\beta < \gamma < \alpha} \{(\eta, \xi) \mid \eta(\gamma) \neq \xi(\gamma)\} \cup \bigcap_{\beta < \gamma < \alpha} \{(\eta, \xi) \mid \eta(\gamma) = \xi(\gamma)\} \right), \\ E_0 &= \bigcup_{\alpha < \kappa} \bigcap_{\alpha < \beta < \kappa} \{(\eta, \xi) \mid \eta(\beta) = \xi(\beta)\}. \\ E_S &= E_S^* \cap E_0. \end{aligned}$$

□

The ideas of the following proofs are simple, but are repeated many times in this article in one way or another.

21 Theorem. For all $S \subset \lim \kappa$, $E_S \not\leq_B \text{id}_{2^\kappa}$ and $E_S^* \leq_B \text{id}_{2^\kappa}$.

Proof. For the first part suppose f is a Borel reduction from E_S to id_{2^κ} . Let η be a function such that η and $\bar{\eta} = 1 - \eta$ are both in $C(f)$ (see Definition 2, page 3). This is possible by Lemma 3, page 3. Then $(\eta, \bar{\eta}) \notin E_S$. Let α be so large that $f(\eta) \upharpoonright \alpha \neq f(\bar{\eta}) \upharpoonright \alpha$ and pick β so that

$$f[N_{\eta \upharpoonright \beta} \cap C(f)] \subset N_{f(\eta) \upharpoonright \alpha}$$

and

$$f[N_{\bar{\eta} \upharpoonright \beta} \cap C(f)] \subset N_{f(\bar{\eta}) \upharpoonright \alpha}.$$

This is possible by the continuity of f on $C(f)$. By Lemma 3 pick now a $\zeta \in 2^{[\beta, \kappa)}$ so that $\eta \upharpoonright \beta \frown \zeta \in C(f)$ and $\bar{\eta} \upharpoonright \beta \frown \zeta \in C(f)$ which provides us with a contradiction, since

$$(\eta \upharpoonright \beta \frown \zeta, \bar{\eta} \upharpoonright \beta \frown \zeta) \in E_S, \text{ but } f(\eta \upharpoonright \beta \frown \zeta) \neq f(\bar{\eta} \upharpoonright \beta \frown \zeta)$$

To prove the second part it is sufficient to construct a reduction from E_S^* to $\text{id}_{\kappa^\kappa}$, since $\text{id}_{\kappa^\kappa}$ and id_{2^κ} are bireducible (see [5]). Let us define an equivalence relation \sim on $2^{<\kappa}$ such that $p \sim q$ if and only if $\text{dom } p = \text{dom } q$ and $p \triangle q$ is eventually constant, i.e. for some $\alpha < \text{dom } p$, $(p \triangle q)(\gamma)$ is the same for all $\gamma \in [\alpha, \text{dom } p)$. Let $s: 2^{<\kappa} \rightarrow \kappa$ be a map such that $p \sim q \iff s(p) = s(q)$. Suppose $\eta \in 2^\kappa$ and let us define $\xi = f(\eta)$ as follows. Let β_γ denote the γ :th element of S and let $\xi(\gamma) = s(\eta \upharpoonright \beta_\gamma)$. Now we have $\eta E_S^* \xi$ if and only if $\eta \upharpoonright \beta_\gamma = \xi \upharpoonright \beta_\gamma$ for all $\gamma \in \kappa$ if and only if $f(\eta) = f(\xi)$. □

22 Corollary. Let $S \subset \kappa$. If $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p , then there is no continuous function $C \rightarrow 2^\kappa$ such that $(\eta, \xi) \in E_S \cap C^2 \iff f(\eta) = f(\xi)$.

Proof. Apply the same proof as for the first part of Theorem 21; take C instead of $C(f)$ and work inside N_p , e.g. instead of $\eta, \bar{\eta}$ take $p \cap \eta, p \cap \bar{\eta}$ for suitable $\eta \in 2^{\text{dom } p, \kappa}$. \square

23 Definition. A set $A \subset \kappa$ does not reflect to an ordinal α , if the set $\alpha \cap A$ is non-stationary in α , i.e. there exists a closed unbounded subset of α outside of $A \cap \alpha$.

24 Theorem. If $\kappa = \lambda^+ > \omega_1$ and \square_μ^κ holds, $\mu \leq \lambda$, then for every stationary $S \subset S_\omega^\kappa$, there exists a set $B_{\text{nr}}^\mu(S) \subset S$ (nr for non-reflecting) such that $B_{\text{nr}}^\mu(S)$ does not reflect to any $\alpha \in S_{\leq \mu}^\kappa \cap S_{\geq \omega_1}^\kappa$ and the sets $\lim C_\alpha$ witness that, where $\langle C_\alpha \mid \alpha \in S_{\leq \mu}^\kappa \rangle$ is the \square_λ -sequence, i.e. $\lim C_\alpha \subset \alpha \setminus B_{\text{nr}}^\mu(S)$ for $\alpha \in S_{\leq \mu}^\kappa \cap S_{\geq \omega_1}^\kappa$. Since $\text{cf}(\alpha) > \omega$, $\lim C_\alpha$ is cub in α .

Proof. This is a well known argument and can be found in [11]. Let $g: S \rightarrow \kappa$ be the function defined by $g(\alpha) = \text{OTP}(C_\alpha)$. By the definition of \square_μ^κ , $\text{OTP}(C_\alpha) < \mu$ for $\alpha \in S_\omega^\kappa$, so for $\alpha > \mu$ we have $g(\alpha) < \alpha$. By Fodor's lemma there exists a stationary $B_{\text{nr}}^\mu(S) \subset S$ such that $\text{OTP}(C_\alpha) = \text{OTP}(C_\beta)$ for all $\alpha, \beta \in B_{\text{nr}}^\mu(S)$. If $\alpha \in \lim C_\beta$, then $C_\alpha = C_\beta \cap \alpha$ and therefore $\text{OTP}(C_\alpha) < \text{OTP}(C_\beta)$. Hence $\lim C_\beta \subset \beta \setminus B_{\text{nr}}^\mu(S)$. \square

25 Definition. Let E_i be equivalence relations on $2^{\kappa \times \{i\}}$ for all $i < \alpha$ where $\alpha < \kappa$. Let $E = \bigotimes_{i < \alpha} E_i$ be an equivalence relation on the space $2^{\kappa \times \alpha}$ such that $(\eta, \xi) \in E$ if and only if for all $i < \alpha$, $(\eta \upharpoonright (\kappa \times \{i\}), \xi \upharpoonright (\kappa \times \{i\})) \in E_i$.

Naturally, if $\alpha = 2$, we denote $\bigotimes_{i < 2} E_i$ by just $E_0 \otimes E_1$ and we constantly identify $2^{\kappa \times \{i\}}$ with 2^κ .

26 Definition. Given equivalence relations E_i on $2^{\kappa \times \{i\}}$ for $i < \alpha < \kappa^+$, let $\bigoplus_{i \in I} E_i$ be an equivalence relation on $\bigcup_{i < \alpha} 2^{\kappa \times \{i\}}$ such that η and ξ are equivalent if and only if for some $i < \alpha$, $\eta, \xi \in 2^{\kappa \times \{i\}}$ and $(\eta, \xi) \in E_i$.

Intuitively the operation \oplus is taking disjoint unions of the equivalence relations. As above, if say $\alpha = 2$, we denote $\bigoplus_{i < 2} E_i$ by just $E_0 \oplus E_1$ and we identify $2^{\kappa \times \{i\}}$ with 2^κ .

27 Theorem. Assume that $\lambda \in \text{reg } \kappa$ and GC_λ -characterization holds for κ .

1. Suppose that $S_1, S_2 \subset S_{\geq \lambda}^\kappa$ and that $(S_2 \setminus S_1) \cap S_\lambda^\kappa$ is stationary. Then the following holds:

(a) $E_{S_1} \not\leq_B E_{S_2}$.

(b) If $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p , then there is no continuous function $C \rightarrow 2^\kappa$ such that $(\eta, \xi) \in E_{S_1} \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2}$.

2. Assume that $\kappa = \lambda^+ > \omega_1$, $\mu \in \text{reg}(\kappa) \setminus \{\omega\}$ and \square_μ^κ holds. Let $S \subset S_\omega^\kappa$ be any stationary set and let $B_{\text{nr}}^\mu(S)$ be the set defined by Theorem 24. Then the following holds:

(a) Suppose that $S_1, S_2 \subset S_\mu^\kappa$, $B \subset B_{\text{nr}}^\mu(S)$ and let $S'_1 = S_1 \cup B$, $S'_2 = S_2 \cup B$. If $(S'_2 \setminus S'_1) \cap S_\mu^\kappa$ is stationary, then $E_{S'_1} \not\leq_B E_{S'_2}$.

(b) Let S_1, S_2, B, S'_1 and S'_2 be as above. If $(S'_2 \setminus S'_1) \cap S_\mu^\kappa$ is stationary, $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p , then there is no continuous function $C \rightarrow 2^\kappa$ such that $(\eta, \xi) \in E_{S'_1} \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S'_2}$.

3. Let $S_1, S_2, A_1, A_2 \subset S_\omega^\kappa$ be either such that $S_2 \setminus S_1$ and $A_2 \setminus S_1$ are stationary or such that $S_2 \setminus A_1$ and $A_2 \setminus A_1$ are stationary. Then the following holds:

- (a) $E_{S_1} \otimes E_{A_1} \not\leq_B E_{S_2} \otimes E_{A_2}$.
- (b) If $C \subset (2^\kappa)^2$ (we identify $2^{\kappa \times 2}$ with $(2^\kappa)^2$) is a set which is co-meager in some $N_r = \{\eta \in (2^\kappa)^2 \mid \eta \restriction \text{dom } r = r\}$, $r \in (2^\alpha)^2$, $\alpha < \kappa$, then there is no continuous function f from $C \cap N_r$ to $(2^\kappa)^2$ such that $(\eta, \xi) \in (E_{S_1} \otimes E_{A_1}) \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2} \otimes E_{A_2}$.

4. Assume that $S_1, S_2, A_2 \subset \kappa$ are such that $A_2 \setminus S_1$ and $S_2 \setminus S_1$ are ω -stationary. Then

- (a) $E_{S_1} \not\leq_B E_{S_2} \otimes E_{A_2}$.
- (b) If $p \in 2^{<\kappa}$ and $C \subset N_p$ is any co-meager subset of N_p , then there is no continuous function $C \rightarrow (2^\kappa)^2$ such that $(\eta, \xi) \in E_{S_1} \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2} \otimes E_{A_2}$.

5. Assume that $S_1, A_1, S_2, A_2 \subset \kappa$ are such that $A_2 \setminus A_1$ is ω -stationary. Then

- (a) $E_{S_1} \otimes E_{A_1} \not\leq_B E_{S_2 \cup A_2}$.
- (b) If $p \in (2^{<\kappa})^2$ and $C \subset N_p$ is any co-meager subset of N_p , then there is no continuous function $C \rightarrow 2^\kappa$ such that $(\eta, \xi) \in (E_{S_1} \otimes E_{A_1}) \cap C^2 \iff (f(\eta), f(\xi)) \in E_{S_2 \cup A_2}$.

Proof. Item 1b of the theorem implies item 1a as well as all (b)-parts imply the corresponding (a)-parts, because if $f: 2^\kappa \rightarrow 2^\kappa$ is a Borel function, then it is continuous on the co-meager set $C(f)$ (see Definition 2). Let us start by proving 1b:

Assume that $S_2 \setminus S_1$ is λ -stationary, $p \in 2^{<\kappa}$, $C \subset N_p$ and assume that $f: C \rightarrow 2^\kappa$ is a continuous function as described in the Theorem. Let us derive a contradiction. Define a strategy for player **II** in the game $\text{GC}_\lambda(\kappa \setminus (S_2 \setminus S_1))$ as follows.

Denote the i :th move of player **I** by α_i and the i :th move of player **II** by β_i . During the game, at the i :th move, $i < \lambda$, player **II** secretly defines functions $p_i^0, p_i^1, q_i^0, q_i^1 \in 2^{<\kappa}$ in such a way that for all i and all $j < i$ we have

- (a) $\text{dom } p_j^0 = \text{dom } p_j^1 = \beta_j$ and $\alpha_j \leq \text{dom } q_{j+1}^0 = \text{dom } q_{j+1}^1 \leq \alpha_j$, and if j is a limit, then $\sup_{i < j} \alpha_i \leq \text{dom } q_j^0 = \text{dom } q_j^1 \leq \beta_j$,
- (b) $p_j^0 \subset p_{j+1}^0$, $p_i^1 \subset p_{i+1}^1$, $q_i^0 \subset q_{i+1}^0$ and $q_i^1 \subset q_{i+1}^1$,
- (c) $f[C \cap N_{p_i^0}] \subset N_{q_i^0}$ and $f[C \cap N_{p_i^1}] \subset N_{q_i^1}$.

Suppose it is i :th move and $i = \gamma + 2k$ for some $k < \omega$ and γ which is either 0 or a limit ordinal, and suppose that the players have picked the sequences $(\alpha_j)_{j \leq i}$ and $(\beta_j)_{j < i}$. Additionally **II** has secretly picked the sequences

$$(p_i^0)_{i < j}, (p_i^1)_{i < j}, (q_i^0)_{i < j}, (q_i^1)_{i < j}$$

which satisfy conditions (a)–(c). Assume first that i is a successor. If q_{i-1}^0 is not $E_{S_2}^*(\text{dom } q_{i-1}^0)$ -equivalent to q_{i-1}^1 , then player **II** plays arbitrarily. Otherwise, to decide her next move, player **II** uses Lemma 3 (page 3) to find $\eta \in 2^{[\beta_{i-1}, \kappa)}$ and $\xi = 1 - \eta$, such that $p_{i-1}^0 \cap \eta \in C$ and $p_{i-1}^1 \cap \xi \in C$.

Then she finds $\beta'_i > \alpha_i$ such that $f(p_{i-1}^0 \frown \eta)(\delta) \neq f(p_{i-1}^1 \frown \xi)(\delta)$ for some $\delta \in [\alpha_i, \beta'_i]$. This is possible since f is a reduction and $(q_{i-1}^0, q_{i-1}^1) \in E_{S_2}^*$. Then she picks $\beta_i > \beta'_i$ so that

$$f[C \cap N_{(p_{i-1}^0 \frown \eta) \upharpoonright \beta_i}] \subset N_{f(p_{i-1}^0 \frown \eta) \upharpoonright \beta'_i}$$

and

$$f[C \cap N_{(p_{i-1}^1 \frown \xi) \upharpoonright \beta_i}] \subset N_{f(p_{i-1}^1 \frown \xi) \upharpoonright \beta'_i}.$$

This choice is possible by the continuity of f . Then she (secretly) sets $p_i^0 = (p_{i-1}^0 \frown \eta) \upharpoonright \beta_i$, $p_i^1 = (p_{i-1}^1 \frown \xi) \upharpoonright \beta_i$, $q_i^0 = f(p_{i-1}^0 \frown \eta) \upharpoonright \beta'_i$ and $q_i^1 = f(p_{i-1}^1 \frown \xi) \upharpoonright \beta'_i$. Note that the new partial functions secretly picked by **II** satisfy conditions (a)–(c).

If i is a limit, then player **II** proceeds as above but instead of p_{i-1}^n she uses $\bigcup_{i' < i} p_{i'}^n$, $n \in \{0, 1\}$, and instead of β_{i-1} she uses $\sup_{i' < i} \beta_{i'}$. If i is 0, then proceed in the same way assuming $p_{-1}^0 = p_{-1}^1 = q_{-1}^0 = q_{-1}^1 = \emptyset$ and $\alpha_{-1} = \beta_{-1} = 0$.

Suppose $i = \gamma + 2k + 1$ where γ is again a limit or zero and $k < \omega$. Then the moves go in the same way, except that she sets $\eta = \xi$ instead of $\eta = 1 - \xi$ and requires $f(p_{i-1}^0 \frown \eta)(\delta) = f(p_{i-1}^1 \frown \xi)(\delta)$ for some $\delta \in [\alpha_{i-1}, \beta'_i]$ instead of $f(p_{i-1}^0 \frown \eta)(\delta) \neq f(p_{i-1}^1 \frown \xi)(\delta)$ for some $\delta \in [\alpha_{i-1}, \beta'_i]$. Denote this strategy by σ .

Since $S_2 \setminus S_1$ is stationary and GC_λ -characterization holds for κ , player **I** is able to play against this strategy such that $\sup_{i < \lambda} \alpha_i \in S_2 \setminus S_1$. Suppose they have played the game to the end, so that player **II** used σ , player **I** has won and they have picked the sequence $\langle \alpha_i, \beta_i \mid i < \lambda \rangle$. Let

$$\alpha_\lambda = \sup_{i < \lambda} \alpha_i = \sup_{i < \lambda} \beta_i = \sup_{i < \lambda} \text{dom } p_i = \sup_{i < \lambda} \text{dom } q_i$$

and

$$p_\lambda^0 = \bigcup_{i < \lambda} p_i^0, \quad p_\lambda^1 = \bigcup_{i < \lambda} p_i^1, \quad q_\lambda^0 = \bigcup_{i < \lambda} q_i^0 \text{ and } q_\lambda^1 = \bigcup_{i < \lambda} q_i^1.$$

By continuity, $p_\lambda^0, p_\lambda^1, q_\lambda^0$ and q_λ^1 satisfy condition (c) above and $\text{dom } p_\lambda^0 = \text{dom } p_\lambda^1 = \text{dom } q_\lambda^0 = \text{dom } q_\lambda^1 = \sup_{i < \lambda} \alpha_i = \sup_{i < \lambda} \beta_i$, so α_λ is well defined.

On one hand q_λ^0 and q_λ^1 cannot be extended in an E_{S_2} -equivalent way, since either they cofinally get same and different values below $\alpha_\lambda \in S_2$, or they are not $E_{S_2}^*(\gamma)$ -equivalent already for some $\gamma < \alpha_\lambda$. On the other hand p_λ^0 and p_λ^1 can be extended in an E_{S_1} -equivalent way, since α_λ is not in S_1 and for all $\gamma < \lambda$, $\sup_{i < \gamma} \alpha_i$ is not μ -cofinal for any $\mu \geq \lambda$, so cannot be in S_1 either (*).

Let $\eta, \xi \in 2^\kappa$ be extensions of p_λ^0 and p_λ^1 respectively such that $(\eta, \xi) \in E_{S_1} \cap C^2$. Now $f(\eta)$ and $f(\xi)$ cannot be E_{S_2} -equivalent, since by condition (c), they must extend q_λ^0 and q_λ^1 respectively.

Now let us prove 2b which implies 2a. Let $\langle C_\alpha^\mu \mid \alpha \in S_{\leq \mu}^\kappa \rangle$ be the \square_μ^κ -sequence and denote by t^μ the function $\alpha \mapsto C_\alpha^\mu$.

Let player **II** define her strategy in the game $\text{GC}(\kappa \setminus (S'_2 \setminus S'_1))$ exactly as in the proof of 1b. Note that $S'_2 \setminus S'_1 = S_2 \setminus S_1$ since $\mu > \omega$. Denote this strategy by σ . We know that, as above, Player **I** is able to beat σ . However, now it is not enough, because in order to be able to extend p_μ^0 and p_μ^1 in an $E_{S'_1}$ -equivalent way, he needs to ensure that

$$S'_1 \cap \lim_\omega(\{\alpha_i \mid i < \mu\}) = \emptyset \quad (**)$$

where $\lim_\omega X$ is the set of ω -limits of elements of X , i.e. we cannot rely on the sentence followed by (*) above. On the other hand (**) is sufficient, because $S'_1 \subset S_\mu^\kappa \cup S_\omega^\kappa$.

Let us show that it is possible for player **I** to play against σ as required.

Let $\nu > \kappa$ be a sufficiently large cardinal and let M be an elementary submodel of $\langle H_\nu, \sigma, \kappa, t^\mu \rangle$ such that $|M| < \kappa$ and $\alpha = \kappa \cap M$ is an ordinal in $S'_2 \setminus S'_1$.

In the game, suppose that the sequence $d = \langle \alpha_j, \beta_j \mid j < i \rangle$ has been played before move i and suppose that this sequence is in M . Player **I** will now pick α_i to be the smallest element in C_α^μ which is above $\sup_{j < i} \beta_j$. Since $C_\alpha^\mu \cap \beta = C_\beta^\mu$ for any $\beta \in \lim C_\alpha^\mu$ and $C_\beta^\mu \in M$, this element is definable in M from the sequence d and t^μ . This guarantees that the sequence obtained on the following move is also in M . At limits the sequence is in M , because it is definable from t^μ and σ . Since $\text{OTP}(C_\alpha^\mu) = \mu$, the game ends at α and player **I** wins. Also the requirement (**) is satisfied because he picked elements only from C_α^μ and so $\lim_\omega \{\alpha_i \mid i < \mu\} \subset \lim_\omega (C_\alpha^\mu) \subset \alpha \setminus B$ which gives the result.

Next let us prove 3b which again implies 3a. The proofs of 4 and 5 are very similar to that of 3 and are left to the reader.

So, let S_1, A_1, S_2, A_2, C and r be as in the statement of 3 and suppose that there is a counter example f . Assume that $S_2 \setminus S_1$ and $A_2 \setminus S_1$ are stationary, the other case being symmetric. Let us define the property P :

P : There exist $p, p' \in (2^\alpha)^2$, $p = (p_1, p_2)$ and $p' = (p'_1, p'_2)$, such that

- (a) $r \subset p \cap p'$,
- (b) $p_2 = p'_2$, $(p_1, p'_1) \in E_{S_1}^*(\alpha + 1)$ (see Definition 19, page 7),
- (c) for all $\eta \in C \cap N_p$ and $\eta' \in C \cap N_{p'}$, $\eta = (\eta_1, \eta_2)$, $\eta' = (\eta'_1, \eta'_2)$, if $\eta_2 = \eta'_2$ and $(\eta_1, \eta'_1) \in E_{S_1}^*$, then $f(\eta)_1 \triangle f(\eta')_1 \subset \text{dom } p_1$ where $f(\eta) = (f(\eta)_1, f(\eta)_2)$.

We will show that both P and $\neg P$ lead to a contradiction. Assume first $\neg P$. Now the argument is similar to the proof of 1b. Player **II** defines her strategy in the same way but this time she chooses the elements p_i^n and q_i^n from $(2^\alpha)^2$ instead of 2^α so that $p_i^n = (p_{i,1}^n, p_{i,2}^n)$, $q_i^n = (q_{i,1}^n, q_{i,2}^n)$ and for all $i < \lambda$, $p_{i,2}^0 = p_{i,2}^1$. In building the strategy she looks only at $q_{i,1}^n$ and ignores $q_{i,2}^n$. In other words she pretends that the game is for E_{S_1} and E_{S_2} in the proof of 1. At the even moves she extends $p_{i,1}^0$ and $p_{i,1}^1$ by η and η' which witness the failure of item (c) (but not of (a) and (b)) of property P for p_i^0 and p_i^1 . Then there is $\alpha \in f(\eta)_1 \triangle f(\eta')_1$, $\alpha > \text{dom } p_{i,1}^0$. And then she chooses $q_{i,1}^0$ and $q_{i,1}^1$ to be initial segments of $f(\eta)_1$ and $f(\eta')_1$ respectively.

At the odd moves she just extends $p_{i,1}^0$ and $p_{i,1}^1$ in an E_{S_1} -equivalent way, so that she finds an $\alpha > \text{dom } p_{i,1}^0$, $q_{i,1}^0$ and $q_{i,1}^1$ such that $q_{i,1}^0(\alpha) = q_{i,1}^1(\alpha)$ and $f[N_{p_i^0} \cap C] \subset N_{q_i^0}$.

As in the proof of 1, **I** responses by playing towards an ordinal in $S_2 \setminus S_1$. During the game they either hit a point at which $q_{i,2}^0$ and $q_{i,2}^1$ cannot be extended to be E_{A_2} -equivalent or else they play the game to the end whence $q_{\lambda,1}^0$ and $q_{\lambda,1}^1$ cannot be extended in a E_{S_2} -equivalent way but p_λ^0 and p_λ^1 can be extended to $E_{S_1} \otimes E_{A_1}$ -equivalent way.

Assume that P holds. Fix p and p' which witness that. Now player **II** builds her strategy as if they were playing between E_{S_1} and E_{A_2} . This time she concentrates on $q_{i,2}^0$ and $q_{i,2}^1$ instead of $q_{i,1}^0$ and $q_{i,1}^1$. At the even moves she extends $p_{i,1}^0$ and $p_{i,1}^1$ by η and $\bar{\eta}$ respectively for some η . Also, as above, $p_{i,2}^0$ and $p_{i,2}^1$ are extended in the same way. By item (c) $f(\eta)_1 \triangle f(\eta')_1$ is bounded by $\text{dom } p_{i,1}^0$, but $f(\eta)$ and $f(\eta')$ can't be $E_{S_2} \otimes E_{A_2}$ -equivalent, because f is assumed

to be a reduction. Hence there must exist $\alpha > \text{dom } p_{i,1}^0, q_{i,2}^0$ and $q_{i,2}^0$ such that $q_{i,2}^0(\alpha) \neq q_{i,2}^1(\alpha)$. The rest of the argument goes similarly as above. \square

28 Corollary. *If GC_λ -characterization holds for κ and $S \subset \kappa$ is λ -stationary, then $E_0 \not\leq E_S$. In particular, if S is ω -stationary, then $E_0 \not\leq E_S$.*

Proof. Follows from Theorem 27.1a by taking $S_1 = \emptyset$, since $E_\emptyset = E_0$ and GC_ω -characterization holds for κ . \square

29 Corollary. *There is an antichain² of Borel equivalence relations on 2^κ of length 2^κ .*

Proof. Take disjoint ω -stationary sets S_i , $i < \kappa$. Let $f: \kappa \times 2 \rightarrow \kappa$ be a bijection. For each $\eta \in 2^\kappa$ let $A_\eta = \{(\alpha, n) \in \kappa \times 2 \mid (n = 0 \wedge \eta(\alpha) = 1) \vee (n = 1 \wedge \eta(\alpha) = 0)\}$. For each $\eta \neq \xi$ clearly $A_\eta \setminus A_\xi \neq \emptyset \neq A_\xi \setminus A_\eta$. Let

$$S_\eta = \bigcup_{i \in f[A_\eta]} S_i.$$

Now $\{E_{S_\eta} \mid \eta \in 2^\kappa\}$ is an antichain by Theorem 27.1b. \square

Let us show that all these relations are below E_0 . It is already shown that they are not above it (Corollary 28), provided GC_λ -characterization holds for κ . Again, similar ideas will be used in the proof of Theorems 11 and 12.

30 Theorem. *For all S , $E_S \leq_B E_0$.*

Proof. Let us show that E_S is reducible to E_0 on κ^κ which is in turn bireducible with E_0 on 2^κ (see [5]). Let us define an equivalence relation \sim on $2^{<\kappa}$ as on page 7, such that $p \sim q$ if and only if $\text{dom } p = \text{dom } q$ and $p \triangle q$ is eventually constant, i.e. for some $\alpha < \text{dom } p$, $(p \triangle q)(\gamma)$ is the same for all $\gamma \in [\alpha, \text{dom } p)$. Let $s: 2^{<\kappa} \rightarrow \kappa$ be a map such that $p \sim q \iff s(p) = s(q)$. Let $\{A_i \mid i \in S\}$ be a partition of $\lim \kappa$ into disjoint unbounded sets. Suppose $\eta \in 2^\kappa$ and define $f(\eta) = \xi \in \kappa^\kappa$ as follows.

- If α is a successor, $\alpha = \beta + 1$, then $\xi(\alpha) = \eta(\beta)$.
- If α is a limit, then $\alpha \in A_i$ for some $i \in S$. Let $\xi(\alpha) = s(\eta \restriction i)$

Let us show that f is the desired reduction from E_S to E_0 . Assume that η and ξ are E_S -equivalent. If α is a limit and $\alpha \in A_i$, then, since η and ξ are E_S -equivalent, we have $\eta \restriction i \sim \xi \restriction i$, so $s(\eta \restriction i) = s(\xi \restriction i)$ and so $f(\eta)(\alpha) = f(\xi)(\alpha)$. There is β such that $\eta(\gamma) = \xi(\gamma)$ for all $\gamma > \beta$. This implies that for all successors $\gamma > \beta$ we also have $f(\eta)(\gamma) = f(\xi)(\gamma)$. Hence $f(\eta)$ and $f(\xi)$ are E_0 -equivalent. Assume now that η and ξ are not E_S -equivalent. Then there are two cases:

1. $\eta \triangle \xi$ is unbounded. Now $f(\eta)(\beta + 1) = \eta(\beta)$ and $f(\xi)(\beta + 1) = \xi(\beta)$ for all β , so we have

$$\{\beta \mid \eta(\beta) \neq \xi(\beta)\} = \{\beta \mid f(\eta)(\beta + 1) \neq f(\xi)(\beta + 1)\}.$$

If the former is unbounded, then so is the latter.

2. For some $i \in S$, $\eta \restriction i \not\sim \xi \restriction i$. This implies that $f(\eta)(\alpha) \neq f(\xi)(\alpha)$ for all $\alpha \in A_i$. and we get that $\{\beta \mid f(\eta)(\beta) \neq f(\xi)(\beta)\}$ is again unbounded.

It is easy to check that f is continuous. \square

²By an antichain I refer here to a family of pairwise incomparable elements unlike e.g. in forcing context.

4.3 Proofs of the Main Theorems

Proof of Theorem 11. The subject of the proof is that for a regular $\lambda < \kappa$, if GC_λ -characterization holds for κ , then the order $\langle \mathcal{P}(\kappa), \subset_{\text{NS}(\lambda)} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$ strictly below E_0 and above id_{2^κ} .

Let $h: \omega \times \kappa \rightarrow \kappa$ be a bijection. Let $\tilde{h}: 2^{\omega \times \kappa} \rightarrow 2^\kappa$ be defined by $\tilde{h}(\eta)(\alpha) = \eta(h^{-1}(\alpha))$. We define the topology on $2^{\omega \times \kappa}$ to be generated by the sets $\{\tilde{h}^{-1}V \mid V \text{ is open in } 2^\kappa\}$. Then \tilde{h} is a homeomorphism between $2^{\omega \times \kappa}$ and 2^κ . If $g: \kappa \times \kappa \rightarrow \kappa$ is a bijection, we similarly get a topology onto $2^{\kappa \times \kappa}$ and a homeomorphism \tilde{g} from $2^{\kappa \times \kappa}$ onto 2^κ . By combining these two we get a homeomorphism between $2^{\omega \times \kappa} \times 2^\kappa$ and 2^κ , and so without loss of generality we can consider equivalence relations on these spaces.

For a given equivalence relation E on 2^κ , let \overline{E} be the equivalence relation on $2^{\omega \times \kappa} \times 2^\kappa$ defined by

$$((\eta, \xi), (\eta', \xi')) \in \overline{E} \iff \eta = \eta' \wedge (\xi, \xi') \in E.$$

Essentially \overline{E} is the same as $\text{id} \otimes E$, since $2^{\omega \times \kappa} \approx 2^\kappa$.

31 Remark. Corollary 22, Theorem 27 and Corollary 28 hold even if E_S is replaced everywhere by $\overline{E_S}$ for all $S \subset \kappa$.

Proof. Let us show this for Theorem 27.1. The proof goes exactly as the proof of Theorem 27.1, but player **I** now picks the functions p_k^n from $\bigcup_{\alpha < \kappa} 2^{\omega \times \alpha} \times 2^\alpha$ instead of $2^{<\kappa}$, $p_k^n = (p_{k,1}^n, p_{k,2}^n)$, and requires that at each move $p_{k,1}^0 = p_{k,1}^1$. Otherwise the argument proceeds in the same manner. Similarly for 27.2, 27.3, 27.4 and 27.5.

Modify the proof of the first part of Theorem 21 in a similar way to obtain the result for Corollary 22. Corollary 28 follows from the modified version of Theorem 27. \square

For $S \subset \kappa$ let

$$G(S) = \overline{E_{S^\kappa \setminus S}}.$$

Let us show that $G: \mathcal{P}(\kappa) \rightarrow \mathcal{E}_\kappa^B$ is the desired embedding. Without loss of generality let us assume that G is restricted to $\mathcal{P}(S_\lambda^\kappa)$, whence stationary is the same as λ -stationary and non-stationary is the same as not λ -stationary. For arbitrary $S_1, S_2 \subset S_\lambda^\kappa$ we have to show:

1. If $S_2 \setminus S_1$ is stationary, then $\overline{E_{S_1}} \not\leq_B \overline{E_{S_2}}$
2. If $S_2 \setminus S_1$ is non-stationary, then $\overline{E_{S_1}} \leq_B \overline{E_{S_2}}$
3. $\text{id}_{2^\kappa} \leq_B \overline{E_{S_1}} \leq_B E_0$.

If $\eta \in 2^{\omega \times \kappa}$, denote $\eta_i(\alpha) = \eta(i, \alpha)$ and $(\eta_i)_{i < \omega} = \eta$.

Claim 1. If $S_2 \setminus S_1$ is stationary, then $\overline{E_{S_1}} \not\leq_B \overline{E_{S_2}}$. Also $E_0 \not\leq \overline{E_S}$.

Proof. Follows from Theorem 27.1a and Remark 31. \square

Claim 2. If $S_2 \setminus S_1$ is non-stationary, then $\overline{E_{S_1}} \leq_B \overline{E_{S_2}}$.

Proof. Let us split this into two parts according to the stationarity of S_2 . Assume first that S_2 is non-stationary. Let C be a cub set outside S_2 . Let $f: 2^\kappa \rightarrow 2^{\omega \times \kappa} \times 2^\kappa$ be the function defined as follows. For $\eta \in 2^\kappa$ let $f(\eta) = \langle (\eta_i)_{i < \omega}, \xi \rangle$ be such that $\eta_i(\alpha) = 0$ for all $\alpha < \kappa$ and $i < \omega$ and $\xi(\alpha) = 0$ for all $\alpha \notin C$. If $\alpha \in C$, then let $\xi(\alpha) = \eta(\text{OTP}(\alpha \cap C))$. This is easily verified to be a reduction from E_0 to $\overline{E_{S_2}}$. By the following Claim 3, $\overline{E_{S_1}} \leq_B E_0$, so we are done.

Assume now that S_2 is stationary. Note that then S_1 is also stationary. Let C be a cub set such that $S_2 \cap C \subset S_1$. Assume that $\langle (\eta_i)_{i < \omega}, \xi \rangle \in 2^{\omega \times \kappa} \times 2^\kappa$ and let us define

$$f(\langle (\eta_i)_{i < \omega}, \xi \rangle) = \langle (\eta'_i)_{i < \omega}, \xi' \rangle \in 2^{\omega \times \kappa} \times 2^\kappa$$

as follows. For $i \geq 0$ let

$$\eta'_{i+1} = \eta_i.$$

For all $\alpha < \kappa$, let $\xi'(\alpha) = \xi(\min(C \setminus \alpha))$. Then let s be the function as defined in the proof of Theorem 21 (on page 7) and for all $\alpha < \kappa$ let $\beta(\alpha)$ be the α :th element of $S_1 \setminus S_2$. For all $\alpha < \kappa$, let

$$\eta'_0(\alpha) = s(\xi \restriction \beta(\alpha)).$$

Let us show that this defines a continuous reduction.

Suppose $\langle (\eta_i^0)_{i < \omega}, \xi^0 \rangle$ and $\langle (\eta_i^1)_{i < \omega}, \xi^1 \rangle$ are $\overline{E_{S_1}}$ -equivalent. Denote their images under f by $\langle (\rho_i^0)_{i < \omega}, \zeta^0 \rangle$ and $\langle (\rho_i^1)_{i < \omega}, \zeta^1 \rangle$ respectively. Since $\eta_i^0 = \eta_i^1$ for all $i < \omega$, we have $\rho_i^0 = \rho_i^1$ for all $0 < i < \omega$. Since for all $\alpha \in S_1$ we have that $\xi^0 \restriction \alpha$ and $\xi^1 \restriction \alpha$ are \sim -equivalent (as in the definition of s), we have that $\rho_0^0(\beta) = \rho_0^1(\beta)$ for all $\beta < \kappa$.

Suppose now that $\alpha \in S_2$. The aim is to show that $\zeta^0 \restriction \alpha \sim \zeta^1 \restriction \alpha$. If $\alpha \notin C$, then there is $\beta < \alpha$ such that $C \cap (\beta, \alpha) = \emptyset$, because C is closed. This implies that for all $\beta < \gamma < \gamma' < \alpha$, $\min(C \setminus \gamma') = \min(C \setminus \gamma)$, so by the definition of f , $\zeta^0(\gamma) = \zeta^0(\gamma')$ and $\zeta^1(\gamma) = \zeta^1(\gamma')$. Now by fixing γ_0 between β and α we deduce that $\zeta^0 \restriction (\beta, \alpha)$ is constant and $\zeta^1 \restriction (\beta, \alpha)$ is constant, since for all $\gamma < \alpha$ we have $\zeta^0(\gamma) = \zeta^0(\gamma_0)$ and $\zeta^1(\gamma) = \zeta^1(\gamma_0) = \zeta^1(\gamma)$. Hence $(\zeta^0 \triangle \zeta^1) \restriction (\beta, \alpha)$ is constant which by the definition of \sim implies that $\zeta^0 \restriction \alpha \sim \zeta^1 \restriction \alpha$.

If $\alpha \in C$, then, since α is also in S_2 , we have by the definition of C that $\alpha \in S_1$. Thus, there is $\beta < \alpha$ such that $(\xi^0 \triangle \xi^1) \restriction (\beta, \alpha)$ is constant which implies that for some $k \in \{0, 1\}$ we have $(\zeta^0 \triangle \zeta^1)(\gamma) = k$ for all $\gamma \in (\beta, \alpha) \cap C$. But if $\gamma \in (\beta, \alpha) \setminus C$, then, again by the definition of f , we have $(\zeta^0 \triangle \zeta^1)(\gamma) = (\zeta^0 \triangle \zeta^1)(\gamma')$ for some $\gamma \in (\beta, \alpha) \cap C$, so $(\zeta^0 \triangle \zeta^1)(\gamma)$ also equals to k .

This shows that ζ^0 and ζ^1 are $E_{S_2}^*$ -equivalent. It remains to show that they are E_0 -equivalent. But since ξ^0 and ξ^1 are E_0 -equivalent, the number $k \in \{0, 1\}$ referred above equals 0 for all α large enough and we are done.

Next let us show that if $\langle (\eta_i^0)_{i < \omega}, \xi^0 \rangle$ and $\langle (\eta_i^1)_{i < \omega}, \xi^1 \rangle$ are not $\overline{E_{S_1}}$ -equivalent, then $\langle (\rho_i^0)_{i < \omega}, \zeta^0 \rangle$ and $\langle (\rho_i^1)_{i < \omega}, \zeta^1 \rangle$ are not $\overline{E_{S_2}}$ -equivalent. This is just reversing implications of the above argument. If $\eta_i^0 \neq \eta_i^1$ for some $i < \omega$, then $\rho_{i+1}^0 \neq \rho_{i+1}^1$, so we can assume that $(\xi^0, \xi^1) \notin E_{S_1}$. If ξ^0 and ξ^1 are not $E_{S_1}^*$ -equivalent, then $\rho^0(\alpha) \neq \rho^1(\alpha)$ for some $\alpha < \kappa$.

The remaining case is that ξ^0 and ξ^1 are $E_{S_1}^*$ -equivalent but not E_0 -equivalent. But then in fact $\xi^0 \triangle \xi^1$ is eventually equal to 1, since otherwise the sets

$$C_1 = \{\alpha \mid \{\beta < \alpha \mid (\xi^0 \triangle \xi^1)(\beta) = 1\} \text{ is unbounded in } \alpha\}$$

and

$$C_2 = \{\alpha \mid \{\beta < \alpha \mid (\xi^0 \triangle \xi^1)(\beta) = 0\} \text{ is unbounded in } \alpha\}$$

are both cub and by the stationarity of S_1 , there exists a point $\alpha \in C_1 \cap C_2 \cap S_1$ which contradicts the fact that ξ_0 and ξ_1 are $E_{S_1}^*$ -equivalent. So $\xi^0 \triangle \xi^1$ is eventually equal to 1 and this finally implies that also ζ^0 and ζ^1 cannot be E_0 -equivalent. \square

Claim 3. Let $S \subset S_\lambda^\kappa$. Then $\text{id} \not\leq_B \overline{E_S} <_B E_0$. If S is stationary, then also $E_0 \not\leq_B \overline{E_S}$.

Proof. If $\eta \in 2^\kappa$, let $\eta_0 = \eta$ and $\eta_i(\alpha) = \xi(\alpha) = 0$ for all $\alpha < \kappa$. Then $\eta \mapsto \langle (\eta_i)_{i < \omega}, \xi \rangle$ defines a reduction from id to $\overline{E_S}$. On the other hand $\overline{E_S}$ is not reducible to id by Remark 31.

Let $u: 2^{\omega \times \kappa} \rightarrow 2^\kappa$ be a reduction from $\text{id}_{2^{\omega \times \kappa}}$ to E_0 . Let $v: 2^\kappa \rightarrow 2^\kappa$ be a reduction from E_S to E_0 which exists by 30. Let $\{A, B\}$ be a partition of κ into two disjoint unbounded subsets. Let $(\eta, \eta') \in 2^{\omega \times \kappa} \times 2^\kappa$ and let us define $\xi = f(\eta, \eta') \in 2^\kappa$. If $\alpha \in A$, then let $\xi(\alpha) = u(\eta)(\text{OTP}(\alpha \cap A))$. If $\alpha \in B$, then let $\xi(\alpha) = v(\eta')(\text{OTP}(\alpha \cap B))$. (See page 3 for notation.)

Now if $((\eta_0, \eta'_0), (\eta_1, \eta'_1)) \in (2^{\omega \times \kappa} \times 2^\kappa)^2$ are $\overline{E_S}$ -equivalent, then both $u(\eta_0) \triangle u(\eta_1)$ and $v(\eta'_0) \triangle v(\eta'_1)$ are eventually equal to zero which clearly implies that $f(\eta_0, \eta'_0) \triangle f(\eta_1, \eta'_1)$ is eventually zero, and so $f(\eta_0, \eta'_0)$ and $f(\eta_1, \eta'_1)$ are E_0 -equivalent. Similarly, if (η_0, η'_0) and (η_1, η'_1) are not $\overline{E_S}$ -equivalent, then either $u(\eta_0) \triangle u(\eta_1)$ or $v(\eta'_0) \triangle v(\eta'_1)$ is not eventually zero, and so $f(\eta_0, \eta'_0)$ and $f(\eta_1, \eta'_1)$ are not E_0 -equivalent.

If S is stationary, then $E_0 \not\leq_B \overline{E_S}$ by Corollary 28 and Remark 31. \square

\square

Proof of Theorem 12. Let us review the statement of the Theorem: assuming $\kappa = \omega_1$, or $\kappa = \lambda^+$ and \square_λ , the partial order $\langle \mathcal{P}(\kappa), \subset_{\text{NS}} \rangle$ can be embedded into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$.

If $\kappa = \omega_1$, then this is just the second part (a special case) of Corollary 17 on page 6 and follows from Theorem 11.

Recall Definition 26 on page 8. Let us see that if $\alpha < \kappa$, then $\bigcup_{i < \alpha} 2^{\kappa \times \{i\}}$ is homeomorphic to 2^κ and so the domains of the forthcoming equivalence relations can be thought without loss of generality to be 2^κ . So fix $\alpha < \kappa$. For all $\beta + 1 < \alpha$ let $\zeta_\beta: \beta + 1 \rightarrow 2$ be the function $\zeta_\beta(\gamma) = 0$ for all $\gamma < \beta$ and $\zeta_\beta(\beta) = 1$ and let $\zeta_\alpha: \alpha \rightarrow 2$ be the constant function with value 0. Clearly $(\zeta_\beta)_{\beta \leq \alpha}$ is a maximal antichain. By rearranging the indexation we can assume that $(\zeta_\beta)_{\beta < \alpha}$ is a maximal antichain. If $\eta \in 2^{\kappa \times \{i\}}$, $i < \alpha$, let $\xi = \eta + i$ be the function with $\text{dom } \xi = [i + 1, \kappa)$ and $\xi(\gamma) = \eta(\text{OTP}(\gamma \setminus i))$ and let

$$f(\eta) = \zeta_i \frown (\eta + i).$$

Then f is a homeomorphism $\bigcup_{i < \alpha} 2^{\kappa \times \{i\}} \rightarrow 2^\kappa$.

Assume $S \subset \kappa$ and let us construct the equivalence relation H_S . Denote for short $r = \text{reg } \kappa$, the set of regular cardinals below κ . Since κ is not inaccessible, $|r| < \kappa$. Let $\{K_\mu \subset S_\omega^\kappa \mid \mu \in r\}$ be a partition of S_ω^κ into disjoint stationary sets. For each $\mu \in r \setminus \{\omega\}$, let $A_\mu = B_{\text{nr}}^\mu(K_\mu)$ be the set given by Theorem 24. Additionally let $\{A_\omega^0, A_\omega^1, A_\omega^2, A_\omega^3\}$ be a partition of K_ω into disjoint stationary sets.

Let

$$\begin{aligned} H_S &= (\text{id}_{2^\kappa} \otimes E_{A_\omega^2 \cup ((S \cap S_\omega^\kappa) \setminus A_\omega^0)} \otimes E_{A_\omega^0}) \\ &\oplus (\text{id}_{2^\kappa} \otimes E_{A_\omega^3 \cup ((S \cap S_\omega^\kappa) \setminus A_\omega^1)} \otimes E_{A_\omega^1}) \\ &\oplus \bigoplus_{\substack{\mu \in r \\ \mu > \omega}} (\text{id}_{2^\kappa} \otimes E_{(S \cap S_\mu^\kappa) \cup A_\mu}). \end{aligned}$$

This might require a bit of explanation. H_S is a disjoint union of the equivalence relations listed in the equation. The final part of the equation lists all the relations obtained by splitting the set S into pieces of fixed uncountable cofinality and coupling them with the non-reflecting ω -stationary sets A_μ . The operation $E \mapsto \text{id}_{2^\kappa} \otimes E$ is the same as the operation $E \mapsto \overline{E}$ in the proof of Theorem 11 above after the identification $2^{\omega \times \kappa} \approx 2^\kappa$. The first two lines of the equation deal with the ω -cofinal part of S . It is trickier, because the “coding sets” A_μ also consist of ω -cofinal ordinals. The way we have built up the relations makes it possible to use Theorem 27 to prove that $S \mapsto H_{\kappa \setminus S}$ is the desired embedding.

In order to make the sequel a bit more readable, let us denote

$$\begin{aligned} \mathcal{B}_\omega^0(S) &= (\text{id}_{2^\kappa} \otimes E_{A_\omega^2 \cup ((S \cap S_\omega^\kappa) \setminus A_\omega^0)} \otimes E_{A_\omega^0}), \\ \mathcal{B}_\omega^1(S) &= (\text{id}_{2^\kappa} \otimes E_{A_\omega^3 \cup ((S \cap S_\omega^\kappa) \setminus A_\omega^1)} \otimes E_{A_\omega^1}), \\ \mathcal{B}_\mu(S) &= (\text{id}_{2^\kappa} \otimes E_{(S \cap S_\mu^\kappa) \cup A_\mu}), \end{aligned}$$

for $\mu \in r \setminus \{\omega\}$. With this notation we have

$$H_S = \mathcal{B}_\omega^0(S) \oplus \mathcal{B}_\omega^1(S) \oplus \bigoplus_{\substack{\mu \in r \\ \mu > \omega}} \mathcal{B}_\mu(S).$$

Let us show that $S \mapsto H_{\kappa \setminus S}$ is an embedding from $\langle \mathcal{P}(\kappa), \subset_{\text{NS}} \rangle$ into $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$. Suppose $S_2 \setminus S_1$ is non-stationary. Then for each $\mu \in r \setminus \{\omega\}$ the set

$$((S_\mu^\kappa \cap S_2) \cup A_\mu) \setminus ((S_\mu^\kappa \cap S_1) \cup A_\mu)$$

is non-stationary as well as are the sets

$$(A_\omega^2 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^0)) \setminus (A_\omega^2 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^0))$$

and

$$(A_\omega^3 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^1)) \setminus (A_\omega^3 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^1))$$

so by Claim 2 of the proof of Theorem 11 (page 13) we have for all $\mu \in r \setminus \{\omega\}$ that

$$(\text{id}_{2^\kappa} \otimes E_{(S_1 \cap S_\mu^\kappa) \cup A_\mu}) \leq_B (\text{id}_{2^\kappa} \otimes E_{(S_2 \cap S_\mu^\kappa) \cup A_\mu}),$$

$$(\text{id}_{2^\kappa} \otimes E_{A_\omega^2 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^0)}) \leq_B (\text{id}_{2^\kappa} \otimes E_{A_\omega^2 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^0)}),$$

and

$$(\text{id}_{2^\kappa} \otimes E_{A_\omega^3 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^1)}) \leq_B (\text{id}_{2^\kappa} \otimes E_{A_\omega^3 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^1)}).$$

Of course this implies that for all $\mu \in r \setminus \{\omega\}$

$$(\text{id}_{2^\kappa} \otimes E_{A_\omega^2 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^0)} \otimes E_{A_\omega^0}) \leq_B (\text{id}_{2^\kappa} \otimes E_{A_\omega^2 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^0)} \otimes E_{A_\omega^0})$$

and that

$$(\text{id}_{2^\kappa} \otimes E_{A_\omega^3 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^1)} \otimes E_{A_\omega^1}) \leq_B (\text{id}_{2^\kappa} \otimes E_{A_\omega^3 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^1)} \otimes E_{A_\omega^1})$$

which precisely means that $\mathcal{B}_\omega^0(S_1) \leq_B \mathcal{B}_\omega^0(S_2)$, $\mathcal{B}_\omega^1(S_1) \leq_B \mathcal{B}_\omega^1(S_2)$ and $\mathcal{B}_\mu(S_1) \leq_B \mathcal{B}_\mu(S_2)$ for all $\mu \in r \setminus \{\omega\}$. Combining these reductions we get a reduction from H_{S_1} to H_{S_2} .

Assume that $S_2 \setminus S_1$ is stationary. We want to show that $H_{S_1} \not\leq_B H_{S_2}$. H_{S_1} is a disjoint union the equivalence relations $\mathcal{B}_\omega^0(S_1)$, $\mathcal{B}_\omega^1(S_1)$ and $\mathcal{B}_\mu(S_1)$ for $\mu \in r \setminus \{\omega\}$. Let us call these equivalence relations *the building blocks of H_{S_1}* and similarly for H_{S_2} .

Each building block of H_{S_1} can be easily reduced to H_{S_1} via inclusion, so it is sufficient to show that there is one block that cannot be reduced to H_{S_2} . We will show that if μ_1 is the least cardinal such that $S_{\mu_1}^\kappa \cap (S_2 \setminus S_1)$ is stationary, then

- that building block is $\mathcal{B}_{\mu_1}(S_1)$, if $\mu_1 > \omega$.
- that building block is either $\mathcal{B}_\omega^0(S_1)$ or $\mathcal{B}_\omega^1(S_1)$, if $\mu_1 = \omega$.

Such a cardinal μ_1 exists because κ is not inaccessible and $|r| < \kappa$.

Suppose that f is a reduction from a building block of H_{S_1} , call it \mathcal{B} , to H_{S_2} . H_{S_2} is a disjoint union of less than κ building blocks whose domains' inverse images decompose $\text{dom } f$ into less than κ disjoint pieces and one of them, say C , is not meager. By the Property of Baire one can find a basic open set U such that $C \cap U$ is co-meager in U . Let $C(f)$ be a co-meager set in which f is continuous. Now $f \upharpoonright (U \cap C \cap C(f))$ is a continuous reduction from \mathcal{B} restricted to $(U \cap C \cap C(f))^2$ to a building block of H_{S_2} . Thus it is sufficient to show that this correctly chosen building block of H_{S_1} is not reducible to any of the building blocks of H_{S_2} on any such $U \cap C \cap C(f)$. This will follow from Theorem 27 and Remark 31 once we go through all the possible cases. So the following Lemma concludes the proof.

32 Lemma. *Assume that $\mu_1 \in r$ is the least cardinal such that $(S_2 \setminus S_1) \cap S_{\mu_1}^\kappa$ is stationary. If $\mu_1 > \omega$, then*

- (i) *for all $\mu_2 > \omega$, $\mathcal{B}_{\mu_1}(S_1) \not\leq_B \mathcal{B}_{\mu_2}(S_2)$,*
- (ii) *$\mathcal{B}_{\mu_1}(S_1) \not\leq_B \mathcal{B}_\omega^0(S_2)$,*
- (iii) *$\mathcal{B}_{\mu_1}(S_1) \not\leq_B \mathcal{B}_\omega^1(S_2)$,*

and if $\mu_1 = \omega$, then

- (i*) *for all $\mu_2 > \omega$, $\mathcal{B}_\omega^0(S_1) \not\leq_B \mathcal{B}_{\mu_2}(S_2)$,*
- (ii*) *for all $\mu_2 > \omega$, $\mathcal{B}_\omega^1(S_1) \not\leq_B \mathcal{B}_{\mu_2}(S_2)$,*
- (iii*) *either*

$$\mathcal{B}_\omega^0(S_1) \not\leq_B \mathcal{B}_\omega^0(S_2) \text{ and } \mathcal{B}_\omega^0(S_1) \not\leq_B \mathcal{B}_\omega^1(S_2) \tag{1}$$

or

$$\mathcal{B}_\omega^1(S_1) \not\leq_B \mathcal{B}_\omega^0(S_2) \text{ and } \mathcal{B}_\omega^1(S_1) \not\leq_B \mathcal{B}_\omega^1(S_2). \tag{2}$$

Proof of the lemma. First we assume $\mu_1 > \omega$.

(i) There are two cases:

Case 1: $\mu_2 = \mu_1$. Denote $B = A_{\mu_1} = A_{\mu_2}$ and $S'_1 = (S_1 \cap S_{\mu_1}^\kappa) \cup B$ and $S'_2 = (S_2 \cap S_{\mu_2}^\kappa) \cup B$. Now $\mathcal{B}_{\mu_1}(S_1) = \text{id} \otimes E_{S'_1}$ and $\mathcal{B}_{\mu_2}(S_2) = \text{id} \otimes E_{S'_2}$. Since by definition $B = B_{\text{nr}}^\mu(K_\mu)$ where $K_\mu \subset S_\omega^\kappa$ is stationary, and $(S_2 \setminus S_1) \cap S_{\mu_1}^\kappa$ is stationary, the sets S'_1 and S'_2 satisfy the assumptions of Theorem 27.2b, so the statement follows from Theorem 27.2b and Remark 31.

Case 2: $\mu_2 \neq \mu_1$. Let $S'_1 = (S_1 \cap S_{\mu_1}^\kappa) \cup A_{\mu_1}$ and $S'_2 = (S_2 \cap S_{\mu_2}^\kappa) \cup A_{\mu_2}$ whence $B_{\mu_1}(S_1) = \text{id} \otimes E_{S'_1}$ and $B_{\mu_2}(S_2) = \text{id} \otimes E_{S'_2}$. Now $S'_1 \subset S_{\omega}^\kappa$ and $S'_2 \subset S_{\omega}^\kappa$ and since $A_{\mu_1} \cap A_{\mu_2} = \emptyset$, the result follows from Theorem 27.1b and Remark 31.

(ii) Let $S'_1 = (S_1 \cap S_{\mu_1}^\kappa) \cup A_{\mu_1}$, $S'_2 = A_\omega^2 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^0)$, and $A'_2 = A_\omega^0$. By definition,

$$B_\omega^0(S_2) = \text{id}_{2^\kappa} \otimes E_{S'_2} \otimes E_{A'_2}$$

and $B_{\mu_1}(S_1) = E_{S'_1}$. Since $A_{\mu_1} \cap A_\omega^2 = \emptyset$, $S'_1 \cap S_\omega^\kappa = A_{\mu_1}$ and $A_\omega^2 \subset S'_2$, we have that $S'_2 \setminus S'_1$ is ω -stationary, because it contains A_ω^2 . Also $A_\omega^0 \setminus S'_1 = A_\omega^0$, because $S'_1 \cap A_\omega^0 = \emptyset$, so $A'_2 \setminus S'_1$ is ω -stationary. Now the result follows from Theorem 27.4b and Remark 31.

(iii) Similar to (ii).

Then we assume $\mu_1 = \omega$.

(i*) Let $S'_1 = A_\omega^2 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^0)$, $A'_1 = A_\omega^0$, $A'_2 = A_{\mu_2}$ and $S'_2 = (S_2 \cap S_{\mu_2}^\kappa)$. Since $A_\omega^0 \cap A_{\mu_2} = \emptyset$, we have that $A'_2 \setminus A'_1$ is ω -stationary, so by Theorem 27.5 and Remark 31,

$$\text{id} \otimes E_{S'_1} \otimes E_{A'_1} \not\leq_B \text{id} \otimes E_{S'_2 \cup A'_2},$$

which by definitions is exactly the subject of the proof.

(ii*) Similar to (i*).

(iii*) The situation is split into two cases, the latter of which is split into two subcases:

Case 1: $((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^2 \cup A_\omega^0)$ is stationary. Let $S'_1 = A_\omega^2 \cup ((S_1 \cap S_\omega^\kappa) \setminus A_\omega^0)$, $A'_1 = A_\omega^0$, $S'_2 = A_\omega^2 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^0)$ and $A'_2 = A_\omega^0$. Now $A'_2 \setminus S'_1$ is obviously ω -stationary, since it is equal to A_ω^0 . Also $S'_2 \setminus S'_1$ is stationary, because it equals to $((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^2 \cup A_\omega^0)$ which is stationary by the assumption. Now the first part of (1) follows from Theorem 27.3b and Remark 31, because $\mathcal{B}_\omega^0(S_1) = \text{id} \otimes E_{S'_1} \otimes E_{A'_1}$ and $\mathcal{B}_\omega^0(S_2) = \text{id} \otimes E_{S'_2} \otimes E_{A'_2}$. On the other hand let $S''_2 = A_\omega^3 \cup ((S_2 \cap S_\omega^\kappa) \setminus A_\omega^1)$ and $A''_2 = A_\omega^1$. Now $S''_2 \setminus A'_1$ is stationary, because $A_\omega^3 \subset S''_2$ but $A_\omega^3 \cap A'_1 = A_\omega^3 \cap A_\omega^0 = \emptyset$. Also $A''_2 \setminus A'_1$ is stationary since $A''_2 \cap A'_1 = A_\omega^1 \cap A_\omega^0 = \emptyset$. Now also the second part of (1) follows from Theorem 27.3b and Remark 31, because $B_1^0(S_1) = \text{id} \otimes E_{S'_1} \otimes E_{A'_1}$ and $B_1^1(S_2) = \text{id} \otimes E_{S''_2} \otimes E_{A''_2}$.

Case 2: $((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^2 \cup A_\omega^0)$ is non-stationary.

Case 2a: $((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^3 \cup A_\omega^1)$ is stationary. Now (2) follows from Theorem 27.3b and Remark 31 in a similar way as (1) followed in Case 1.

Case 2b: $((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^3 \cup A_\omega^1)$ is non-stationary. Now we have both:

$$((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^2 \cup A_\omega^0) \text{ is non-stationary} \quad (*)$$

and

$$((S_2 \setminus S_1) \cap S_\omega^\kappa) \setminus (A_\omega^3 \cup A_\omega^1) \text{ is non-stationary.} \quad (**)$$

Now from $(*)$ it follows that $S_2 \setminus S_1 \subset_{\text{NS}(\omega)} A_\omega^2 \cup A_\omega^0$. From $(**)$ it follows that $S_2 \setminus S_1 \subset_{\text{NS}(\omega)} A_\omega^3 \cup A_\omega^1$. This is a contradiction, because $S_2 \setminus S_1$ is ω -stationary and $(A_\omega^2 \cup A_\omega^0) \cap (A_\omega^3 \cup A_\omega^1) = \emptyset$.

□

□

5 On Chains In $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$

There are chains of order type κ^+ in Borel equivalence relation on 2^κ :

33 Theorem. *Let $\kappa > \omega$. There are equivalence relations $R_i \in \mathcal{E}_\kappa^B$, for $i < \kappa^+$, such that $i < j \iff R_i \leq_B R_j \leq E_0$.*

34 Remark. In many cases there are κ^+ -long chains in the power set of κ ordered by inclusion modulo the non-stationary ideal whence a weak version of this theorem could be proved using Theorem 12. Namely if the ideal I_{NS}^κ of non-stationary subsets of κ is *not* κ^+ -saturated, then there are κ^+ -long chains. In this case being *not* κ^+ -saturation means that there exists a sequence $\langle A_i \mid i < \kappa^+ \rangle$ of subsets of κ such that A_i is stationary for all i but $A_i \cap A_j$ is non-stationary for all $i \neq j$. Now let f_α be a bijection from κ to α for all $\alpha < \kappa^+$ and let

$$B_\alpha = \bigcap_{i < \alpha} A_i = \{ \alpha \mid \text{for some } i < \alpha, \alpha \in A_{f_\alpha(i)} \}$$

It is not difficult to see that $\langle B_\alpha \mid \alpha < \kappa^+ \rangle$ is a chain. On the other hand the existence of such a chain implies that I_{NS}^κ is not κ^+ -saturated.

By a theorem of Gitik and Shelah [11, Theorem 23.17], I_{NS}^κ is not κ^+ -saturated for all $\kappa \geq \aleph_2$. By a result of Shelah [11, Theorem 38.1], it is consistent relative to the consistency of a Woodin cardinal that $I_{\text{NS}}^{\aleph_1}$ is \aleph_2 -saturated in which case there are no chains of length ω_2 in $\langle \mathcal{P}(\omega_1), \subset_{\text{NS}} \rangle$. On the other hand in the model provided by Shelah, CH fails. According to Jech [3] it is an open question whether CH implies that $I_{\text{NS}}^{\aleph_1}$ is not \aleph_2 -saturated.

However, as the following shows, it follows from ZFC that there are κ^+ -long chains in $\langle \mathcal{E}_\kappa^B, \leq_B \rangle$ for any uncountable κ .

Proof of Theorem 33. By the proof of Corollary 29, page 12, one can find ω -stationary sets S_i for $i < \kappa^+$ such that $S_i \setminus S_j$ and $S_j \setminus S_i$ are stationary whenever $i \neq j$. For all $j \in [1, \kappa^+)$, let

$$R_j = \bigoplus_{i < j} E_{S_i},$$

where the operation \oplus is from Definition 26, page 8.

Let us denote $P_A = \bigcup_{i \in A} 2^{\kappa \times \{i\}}$ for $A \subset \kappa^+$, i.e. for example $P_j = \bigcup_{i < j} 2^{\kappa \times \{i\}}$.

Let us show that

1. if $i < j$, then $R_i \leq_B R_j$,
2. if $i < j$, then $R_j \not\leq_B R_i$,
3. for all $i < \kappa^+$, $R_i \not\leq_B E_0$.

Item 1 is simple: let $f: P_i \rightarrow P_j$ be the inclusion map (as $P_i \subset P_j$). Then f is clearly a reduction from R_i to R_j .

Suppose then that $i < j$ and that $i \leq k < j$. To prove 2 it is sufficient to show that there is no reduction from E_{S_k} to R_j . Let us assume that $f: 2^\kappa \rightarrow P_j$ is a Borel reduction from E_{S_k} to R_j . Now

$$2^\kappa = \bigcup_{\alpha < i} f^{-1}[P_{\{\alpha\}}],$$

so one of the sets $f^{-1}[P_{\{\alpha\}}]$ is not meager; let α_0 be an index witnessing this. Note that $\alpha_0 < k$, because $\alpha_0 < i \leq k$. Because f is a Borel function and Borel sets have the Property of Baire, we can find a $p \in 2^{<\kappa}$ such that $C = N_p \cap C(f) \cap f^{-1}[P_{\{j\}}]$ is co-meager in N_p . But now $f \upharpoonright C$ is a continuous reduction from $E_{S_k} \cap C^2$ to E_{S_α} which contradicts Theorem 27.1b.

To prove 3 we will show first that $R_i \leq_B \bigoplus_{j < i} E_0$ and then that $\bigoplus_{j < i} E_0 \leq_B E_0$, after which we will show that $E_0 \not\leq_B R_i$ for all i .

Let f_j be a reduction from E_{S_j} to E_0 for all $j < i$ given by Claim 3 of the proof of Theorem 11. Then combine these reductions to get a reduction from R_i to $\bigoplus_{j < i} E_0$. To be more precise, for each $\eta \in P_{\{j\}}$ let $f(\eta)$ be ξ such that $\xi \in P_{\{j\}}$ and $\xi = f_j(\eta)$.

Let $\{A_k \mid k \leq i\}$ be a partition of κ into disjoint unbounded sets. Let $\eta \in P_i$. By definition, $\eta \in P_{\{k\}}$ for some $k < i$. Define $\xi = F(\eta)$ as follows. Let $f: A_i \rightarrow \kappa$ be a bijection.

- If $\alpha \in A_i$, then let $\xi(\alpha) = \eta(f(\alpha))$.
- If $\alpha \in A_j$ and $j \neq k$, then let $\xi(\alpha) = 0$.
- If $\alpha \in A_k$, then let $\xi(\alpha) = 1$.

It is easy to see that F is a continuous reduction.

Assume for a contradiction that $E_0 \leq_B R_i$ for some $i < \kappa^+$. Then by 1 and transitivity, $E_0 \leq_B R_j$ for all $j \in [i, \kappa^+)$. By the above also $R_j \leq_B E_0$ for all $j \in [i, \kappa^+)$ which, again by transitivity, implies that the relations R_j for $j \in [i, \kappa^+)$ are mutually bireducible to each other which contradicts 2. \square

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